Ma2a Practical – Recitation 3

Fall 2024

Exercise 1. (Gompertz equation)
Consider the equation

$$\frac{dy}{dt} = ry\log\left(\frac{K}{y}\right)$$

where r and K are positive constants.

1. Sketch the graph of $f(y) = ry\log(\frac{K}{y})$ versus y, find the critical points ($f(y) = 0$), and determine whether each equilibrium is asymptotically stable or unstable. Sketch typical solution curves in the extended phase space.

2. Solve the equation with initial condition $y(0) = y_0 > 0$ (you may use the change of variable $u = \log(\frac{y}{K})$).

Exercise 2. (Exact equations) Determine whether each of the following equation is exact, and if it is exact find the solutions.

1. $\frac{y}{x} + 6x + (\ln x - 2)y' = 0.$

2. $e^x \sin y - 2y\sin x + (e^x \cos y + 2\cos x)y' = 0.$

Exercise 3. (Separable equations) Consider the differential equation:

$$\frac{dy}{dx} = 2\sqrt{|y|}, \quad y(0) = 0.$$

Solve the general solution for this differential equation and prove it doesn't satisfy uniqueness theorem.

Solution 1

1. We have the autonomous equation $y' = f(y)$ with $f(y) = ry\log(\frac{K}{y})$.

   (a) Exitence and uniqueness for the IVP. The function f is differentiable and its derivative is continuous on $\mathbb{R}_{>0}$ so we have the existence and uniqueness of solution for any initial value $y(t_0) = y_0 > 0$.

   (b) Equilibrium. Recall that the equilibrium corresponds to constant solutions to the equation. Equivalently, they are zeros of f. Thus, for this equation, there is only one equilibrium $y_0 = K$.

   (c) Stability of the equilibrium. Recall that the equilibrium y_0 is called *stable* if solutions around y_0 go back to y_0 as t increases. If the solutions around y_0 move away from y_0, the equilibrium is called *unstable*. Crucially, stability can be read off from the phase diagram by looking at the sign of the derivative around y_0.

   From the above graph, we see $y' > 0$ for $y < y_0$ and $y' < 0$ for $y > y_0$ so the system tends to go back to y_0, which implys that y_0 is stable.

2. By using the substitution $u = \log\left(\frac{y}{K}\right)$, we have $u' = \frac{Ky'}{y}$ and the equation becomes

$$u' + Kru = 0.$$

   The general solution is $u(t) = Ae^{-Krt}$, and the initial condition $u(0) = \log\left(\frac{y_0}{K}\right)$ gives the value of A. Going back to y, we obtain that the unique solution to the IVP is

$$y(t) = K\exp\left(\log\left(\frac{y_0}{K}\right)e^{-Krt}\right).$$

Solution 2
Both equations are exact. We have:

1. $\frac{d}{dy}(y/x + 6x) = 1/x$.
   $\frac{d}{dx}(\ln x - 2) = 1/x$.

2. $\frac{d}{dy}(e^x \sin(y) - 2y\sin(x)) = e^x \cos(y) - 2\sin(x)$.
   $\frac{d}{dx}(e^x \cos(y) + 2\cos(x)) = e^x \cos(y) - 2\sin(x)$.

1. Solve:
   Given $F_x = y/x + 6x$, we know:
   i) $F = y\ln(x) + 3x^2 + g(y)$.
   We know also that:
   ii) $F_y = \ln x - 2$.
   Differentiating the former w.r.t y, we have $F_y = \ln x + g'(y)$.
   Setting equal i) and ii):
   $g'(y) = -2$, so $g(y) = -2y + c_1$.
   Using this, we have $F = y\ln x + 3x^2 - 2y + c_1$.
   Since we have $\frac{d}{dx}(F(x, y(x))) = 0$, $F(x, y(x))$ must be some constant, c_2.
   So, we have our implicit solution: $y\ln x + 3x^2 - 2y + c_1 = c_2$.
   Condensing constants, we have simply: $y\ln x + 3x^2 - 2y = c$.
   This gives us: $y(x) = \frac{c - 3x^2}{\ln(x) - 2}$.

2. Solve:
   Given $F_x = e^x \sin(y) - 2y\sin(x)$, we know:
   $F = e^x \sin(y) + 2y\cos(x) + g(y)$.
   We know from this that:
   i) $F_y = e^x \cos(y) + 2\cos(x) + g'(y)$.
   We also know from the original equation that:
   ii) $F_y = e^x \cos(y) + 2\cos(x)$
   We can deduce from i) and ii) that $g'(y) = 0$, which means that $g(y) = c_1$
   Using the same logic as in the previous problem, we have:
   $F = e^x \sin(y) + 2y\cos(x) + c_1 = c_2$.
   Rearranging the constants yields our implicit solution: $e^x \sin(y) + 2y\cos(x) = c$.

Solution 3 Again, this is a separable equation. For $y \geqslant 0$:

$$\frac{dy}{2\sqrt{y}} = dx$$

$$\int \frac{1}{2\sqrt{y}}\, dy = \int 1\, dx$$

$$\sqrt{y} = x + C$$

Squaring both sides gives:

$$y(x) = (x + C)^2$$

Using the initial condition $y(0) = 0$:

$$0 = (0 + C)^2$$

Thus, $C = 0$, and the solution is:

$$y(x) = x^2$$

But $y(x) = 0$ is also a solution since:

$$\frac{d}{dx}(0) = 2\sqrt{|0|} = 0$$

Thus, there are two solutions:
- $y(x) = x^2$, $y(x) = 0$

This doesn't contradict the uniqueness theorem since the assumption of the continuity theorem is not satisfied.